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Approximation of eigenfunctions of elliptic differential operators

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Abstract

Eigenfunctions of elliptic boundary value problems can be well approximated by entire functions of exponential type and, as a consequence, it is possible to transfer approximation results with entire functions to eigenfunction expansions. Here, in particular, we consider Jackson and Bernstein type theorems.

Keywords: Eigenfunction expansions; Bernstein and Jackson theorems

Let us consider the Dirichlet problem for the Laplace operator $-\sum_{j=1}^{n} \partial^2/\partial x_j^2$ on a bounded open domain Ω in $\mathbb{R}^n$. An eigenfunction $\phi_\lambda(x)$ associated to the eigenvalue λ^2 is a non-zero solution to the problem

$$\begin{cases} -\sum_{j=1}^{n} \dfrac{\partial^2}{\partial x_j^2} \phi_\lambda(x) = \lambda^2 \phi_\lambda(x) & \text{if } x \in \Omega, \\ \phi_\lambda(x) = 0 & \text{if } x \in \partial\Omega. \end{cases}$$

There exists a positive sequence $\{\lambda^2\}$ of eigenvalues and an orthonormal system of eigenfunctions $\{\phi_\lambda(x)\}$ complete in $\mathbb{L}^2(\Omega)$ and one can define a Fourier transform

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and Fourier series,

$$\mathscr{F}f(\lambda) = \int_{\Omega} f(y)\overline{\phi_\lambda(y)}\,dy, \qquad f(x) = \sum_{\lambda} \mathscr{F}f(\lambda)\phi_\lambda(x).$$

Similarly the exponentials $\{\exp(2\pi i\xi y)\}$ are eigenfunctions of the Laplace operator on $\mathbb{R}^n$ with eigenvalues $\{4\pi^2|\xi|^2\}$ and in this case the eigenfunction expansion coincides with the classical trigonometric Fourier expansion,

$$\mathbb{F}f(\xi) = \int_{\mathbb{R}^n} f(y)\exp(-2\pi i\xi y)\,dy, \qquad f(x) = \int_{\mathbb{R}^n} \mathbb{F}f(\xi)\exp(2\pi i\xi x)\,d\xi.$$

Since functions on Ω can be extended to all $\mathbb{R}^n$ by putting them equal to zero outside Ω, it is natural to compare these two expansions. Indeed it turns out that they are similar, at least away from the boundary $\partial\Omega$. Recall that, by the Paley–Wiener–Schwartz theorems, the distributions with $\mathbb{F}f(\xi) = 0$ if $|\xi| > T$ coincide with the distributions which are entire functions of exponential type $2\pi T$. By analogy one can define linear combination of eigenfunctions of type Λ by the condition $\mathscr{F}f(\lambda) = 0$ if $\lambda > \Lambda$. Then, roughly speaking, our main result is that strictly inside Ω finite linear combinations of eigenfunctions of type Λ can be arbitrarily well approximated by entire functions of exponential type 2Λ and vice versa. As a consequence, in order to approximate a given function by linear combinations of eigenfunctions it suffices to approximate the function by entire functions of exponential type and then approximate the entire functions by eigenfunctions. In this way, one easily obtains analogous of the Jackson and Bernstein theorems, the degree of approximation of a function by linear combinations of eigenfunctions is controlled by the modulus of continuity and vice versa. The main tool used in these approximations is the synthesis of certain operators by fundamental solutions to the wave equation. This approach has been introduced in the study of the spectral properties of elliptic operators on manifolds and by now it is quite classical.

Approximation theorems for expansions in eigenfunctions of the Laplace operator on manifolds with a rich structure, such as Lie groups or symmetric spaces, are already known. In this case the eigenfunctions are special functions and with convolutions one can construct approximations. On the contrary, for general domains one cannot rely on explicit expressions for eigenfunctions and, since there are no translations, there is no convolution. However, natural substitutes for convolutions are operators defined by spectral analysis. Given a function $m(\lambda)$ on $\mathbb{R}_+$, one can define the operator $m(\sqrt{\Delta})$ via spectral decomposition by

$$m(\sqrt{\Delta})f(x) = \sum_{\lambda} m(\lambda)\mathscr{F}f(\lambda)\phi_\lambda(x).$$

Then, by taking the cosine expansion of $m(\lambda)$ one obtains

$$m(\sqrt{\Delta})f(x) = \pi\int_0^{+\infty} \hat{m}(t)\cos(t\sqrt{\Delta})f(x)\,dt.$$

Hence the operator $m(\sqrt{\Delta})$ can be synthesized by means of fundamental solutions $\cos(t\sqrt{\Delta})$ to the wave equation $\partial^2/\partial t^2 = \Delta$. Now the basic observation is that, by finite propagation of waves, at least for small times and away from the boundary $\partial\Omega$ the solutions to the wave equation on Ω and on $\mathbb{R}^n$ coincide. This allows to compare the operator $m(\sqrt{\Delta})$ on Ω with the corresponding one on $\mathbb{R}^n$. In particular, we shall see that if $m(\lambda)$ is smooth, with $m(\lambda) = 1$ if $\lambda \leqslant 1/2$ and $m(\lambda) = 0$ if $\lambda \geqslant 1$, then $m(\sqrt{\Delta}/\Lambda)$ is an approximation of unity by eigenfunctions of type Λ. Operators of this type will be our basic tool in constructing approximations.

In this paper we consider eigenfunctions of the Laplace operator with Dirichlet boundary conditions, but results and proofs for other boundary conditions are analogous as soon as we stay away from the boundary. The paper is essentially self-contained and we want to mention only a few references. As general references on approximation, and in particular the classical Jackson and Bernstein theorems on Euclidean spaces, see e.g. [3,7]. For approximation by eigenfunctions of the Laplace operator on Lie groups and symmetric spaces see e.g. [2,4] and for approximation on manifolds see the survey [6]. However, it seems to us that the results in these papers do not overlap with ours and, in any case, the techniques used are quite different. For a wave equation approach to the study of spectral properties of differential operators and to the harmonic analysis on manifolds see e.g. [5, XVII]; [9, XII]. See also [1,8] for a wave equation approach to the convergence of Fourier integrals.

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1. Approximation of eigenfunctions

We recall that the Sobolev space $\mathbb{W}_h^2(\mathbb{R}^n)$ and $\mathbb{W}_h^2(\Omega)$, or more generally $\mathbb{W}_h^2(A)$, can be defined via the norms

$$||f(x)||_{\mathbb{W}_h^2(\mathbb{R}^n)} = \left\{ \int_{\mathbb{R}^n} (1 + 4\pi^2|\xi|^2)^h |\mathbb{F}f(\xi)|^2 \, d\xi \right\}^{1/2},$$

$$||f(x)||_{\mathbb{W}_h^2(\Omega)} = \left\{ \sum_{\lambda} (1 + \lambda^2)^h |\mathscr{F}f(\lambda)|^2 \right\}^{1/2},$$

$$||f(x)||_{\mathbb{W}_h^2(A)} = \sum_{|\alpha| \leqslant h} \left\{ \int_A \left| \frac{\partial^\alpha}{\partial x^\alpha} f(x) \right|^2 dx \right\}^{1/2}.$$

In the definition of $\mathbb{W}_h^2(A)$ the index h is a non-negative integer, but in $\mathbb{W}_h^2(\Omega)$ and $\mathbb{W}_h^2(\mathbb{R}^n)$ one can allow $-\infty < h < +\infty$. It is well known that if $\bar{A} \subset \Omega$, then $\mathbb{W}_h^2(\Omega)$ and $\mathbb{W}_h^2(\mathbb{R}^n)$ are continuously imbedded in $\mathbb{W}_h^2(A)$. Also, under suitable assumptions on the domain Ω the classical Sobolev imbeddings hold, in particular functions in $\mathbb{W}_h^2(\Omega)$ and $\mathbb{W}_h^2(\mathbb{R}^n)$ with $h > n/2$ are Lipschitz continuous of order $h - n/2$ in Ω and $\mathbb{R}^n$. Moreover, $\mathbb{L}^1(\Omega)$ and $\mathbb{L}^1(\mathbb{R}^n)$ are contained in $\mathbb{W}_h^2(\Omega)$ and $\mathbb{W}_h^2(\mathbb{R}^n)$ if $h < -n/2$.

However, for what follows we need a milder requirement, namely we assume that $\mathbb{L}^1(\Omega)$ is contained in some of the spaces $\mathbb{W}^2_h(\Omega)$, any h suffices.

Now let $m(s)$ be a smooth bounded even function on $\mathbb{R}$ with compact support, or rapidly decreasing and with rapidly decreasing derivatives. Define $m_\Lambda(s) = m(s/\Lambda)$ and

$$\mathcal{M}_\Lambda f(x) = \sum_\lambda m_\Lambda(\lambda)\mathcal{F}f(\lambda)\phi_\lambda(x).$$

In what follows $m(0) = 1$ and $\Lambda \to \infty$, so that at the limit one formally recover the Fourier expansion of the function. Since the multiplier $m(s)$ decreases rapidly, the operator $\mathcal{M}_\Lambda$ is well defined on integrable functions on Ω. Extend such functions to all $\mathbb{R}^n$ by putting them equal to zero outside Ω and define

$$\mathbb{M}_\Lambda f(x) = \int_{\mathbb{R}^n} m_\Lambda(2\pi|\xi|)\mathbb{F}f(\xi)\exp(2\pi i\xi x)\,d\xi.$$

Since $\mathbb{M}_\Lambda$ is a convolution operator with a smooth fast decaying kernel $K_\Lambda(y) = \Lambda^n K(\Lambda y)$, this operator enjoys many nice properties and in particular it is bounded on the most common function spaces. $\mathcal{M}_\Lambda f(x)$ and $\mathbb{M}_\Lambda f(x)$ are expansions in eigenfunctions of the Laplace operator in Ω and $\mathbb{R}^n$, respectively. The main result of this section is that away from $\partial\Omega$ these two expansions are similar and the Sobolev norms of $\mathcal{M}_\Lambda f(x) - \mathbb{M}_\Lambda f(x)$ are small. Thus, the nice properties of $\mathbb{M}_\Lambda$ are inherited by $\mathcal{M}_\Lambda$.

Theorem 1.1. *Let A be an open set with compact closure in Ω. Then for every non-negative integers h and k there exists c such that*

$$||\mathcal{M}_\Lambda f(x) - \mathbb{M}_\Lambda f(x)||_{\mathbb{W}^2_h(A)} \leqslant c\Lambda^{-k}\int_\Omega |f(y)|\,dy.$$

Proof. In what follows, we may assume that $\Lambda \to +\infty$. Let $\psi(s)$ be a smooth even test function with Fourier cosine transform

$$\hat{\psi}(t) = \frac{2}{\pi}\int_0^{+\infty} \psi(s)\cos(ts)\,ds$$

vanishing for $|t| \geqslant distance(A, \partial\Omega)$. Also assume that $\int_{-\infty}^{+\infty}\psi(s)\,ds = 1$ and $\int_{-\infty}^{+\infty}\psi(s)s^j\,ds = 0$ for $j = 1, 2, \ldots$. Beside the operators $\mathcal{M}_\Lambda$ and $\mathbb{M}_\Lambda$ associated to the multiplier $m_\Lambda(s)$, we also consider the operators $\mathcal{N}_\Lambda$ and $\mathbb{N}_\Lambda$ associated to the regularized multiplier $m_\Lambda * \psi(s) = \int_{-\infty}^{+\infty} m_\Lambda(s-r)\psi(r)\,dr$. We have

$$\begin{aligned}&\mathcal{M}_\Lambda f(x) - \mathbb{M}_\Lambda f(x)\\ &\quad = (\mathcal{M}_\Lambda f(x) - \mathcal{N}_\Lambda f(x)) + (\mathcal{N}_\Lambda f(x) - \mathbb{N}_\Lambda f(x)) + (\mathbb{N}_\Lambda f(x) - \mathbb{M}_\Lambda f(x)).\end{aligned}$$

In order to complete the proof it suffices to show that if x is in A then $\mathcal{N}_\Lambda f(x) = \mathbb{N}_\Lambda f(x)$, while $\mathcal{M}_\Lambda f(x) - \mathcal{N}_\Lambda f(x)$ and $\mathbb{M}_\Lambda f(x) - \mathbb{N}_\Lambda f(x)$ are small. □

Lemma 1.2. *For every x in A we have $\mathcal{N}_\Lambda f(x) = \mathbb{N}_\Lambda f(x)$.*

Proof. We synthesize both operators by means of fundamental solutions to the wave equation:

$$\begin{aligned}
&\sum_{\lambda} m_{\Lambda} * \psi(\lambda)\mathscr{F}f(\lambda)\phi_{\lambda}(x) \\
&\quad = \sum_{\lambda}\left(\int_0^{+\infty} (m_{\Lambda} * \psi)^{\wedge}(t)\cos(\lambda t)\,dt\right)\mathscr{F}f(\lambda)\phi_{\lambda}(x) \\
&\quad = \int_0^{+\infty} \pi\,\widehat{m_{\Lambda}}(t)\,\widehat{\psi}(t)\left(\sum_{\lambda}\cos(\lambda t)\mathscr{F}f(\lambda)\phi_{\lambda}(x)\right)dt \\
&\quad = \pi\int_0^{+\infty} \widehat{m_{\Lambda}}(t)\,\widehat{\psi}(t)\cos(t\sqrt{\Delta})f(x)\,dt,
\end{aligned}$$

where $\cos(t\sqrt{\Delta})f(x)$ solves the Cauchy problem for the wave equation in $\mathbb{R} \times \Omega$,

$$\begin{cases}
\dfrac{\partial^2}{\partial t^2}u(t,x) - \displaystyle\sum_{j=1}^{n}\dfrac{\partial^2}{\partial x_j^2}u(t,x) = 0 & \text{if } t \in \mathbb{R} \text{ and } x \in \Omega, \\
u(t,x) = 0 & \text{if } t \in \mathbb{R} \text{ and } x \in \partial\Omega, \\
u(0,x) = f(x) & \text{if } x \in \Omega, \\
\dfrac{\partial}{\partial t}u(0,x) = 0 & \text{if } x \in \Omega.
\end{cases}$$

Similarly

$$\begin{aligned}
&\int_{\mathbb{R}^n} m_{\Lambda} * \psi(2\pi|\xi|)\mathbb{F}f(\xi)\exp(2\pi i\xi \cdot x)\,d\xi \\
&\quad = \pi\int_0^{+\infty} \widehat{m_{\Lambda}}(t)\,\widehat{\psi}(t)\cos(t\sqrt{\Delta})f(x)\,dt,
\end{aligned}$$

where this time $\cos(t\sqrt{\Delta})f(x)$ solves the wave equation in $\mathbb{R} \times \mathbb{R}^n$,

$$\begin{cases}
\dfrac{\partial^2}{\partial t^2}u(t,x) - \displaystyle\sum_{j=1}^{n}\dfrac{\partial^2}{\partial x_j^2}u(t,x) = 0 & \text{if } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n, \\
u(0,x) = f(x) & \text{if } x \in \mathbb{R}^n, \\
\dfrac{\partial}{\partial t}u(0,x) = 0 & \text{if } x \in \mathbb{R}^n.
\end{cases}$$

Since waves propagate with finite speed and $\cos(t\sqrt{\Delta})f(x)$ depends only upon values $f(y)$ with $|y - x| \leqslant t$, for x away from the boundary $\partial\Omega$ and small time t the solutions to the wave equation in $\mathbb{R}^n$ and Ω coincide. Since $\widehat{\psi}(t) = 0$ if $t \geqslant \mathit{distance}(A, \partial\Omega)$, the lemma follows. □

Lemma 1.3. *For every h and k,*

$$||\mathscr{M}_\Lambda f(x) - \mathscr{N}_\Lambda f(x)||_{\mathbb{W}_h^2(\Omega)} \leqslant c\Lambda^{-k} \int_\Omega |f(y)|\,dy,$$

$$||\mathbb{M}_\Lambda f(x) - \mathbb{N}_\Lambda f(x)||_{\mathbb{W}_h^2(\mathbb{R}^n)} \leqslant c\Lambda^{-k} \int_{\mathbb{R}^n} |f(y)|\,dy.$$

Proof. In order to estimate the Fourier coefficients of $\mathscr{M}_\Lambda f(x) - \mathscr{N}_\Lambda f(x)$, we write

$$\begin{aligned}
|m_\Lambda * \psi(s) - m_\Lambda(s)| &= \left| \int_{-\infty}^{+\infty} (m_\Lambda(s-r) - \sum_{j=0}^{k-1} m_\Lambda^{(j)}(s)(-r)^j/j!)\psi(r)\,dr \right| \\
&\leqslant \frac{\Lambda^{-k}}{k!} \int_{-\infty}^{+\infty} |r|^k |\psi(r)|\,|m^{(k)}((s-\theta r)/\Lambda)|\,dr \\
&\leqslant \frac{\Lambda^{-k}}{k!} \sup_{|r|\leqslant|s|/2,} |m^{(k)}((s-r)/\Lambda)| \int_{-\infty}^{+\infty} |r|^k|\psi(r)|\,dr \\
&\quad + \frac{\Lambda^{-k}}{k!} \sup_{-\infty<s<+\infty} |m^{(k)}(s)| \int_{\{|r|>|s|/2\}} |r|^k |\psi(r)|\,dr.
\end{aligned}$$

Since both $m^{(k)}(s)$ and $\psi(s)$ are rapidly decreasing, for every q we have

$$\sup_{|r|\leqslant|s|/2,} |m^{(k)}((s-r)/\Lambda)| \leqslant c(1+s/\Lambda)^2)^{-q/2} \leqslant c\Lambda^q(1+s^2)^{-q/2},$$

$$\int_{\{|r|>|s|/2\}} |r|^k|\psi(r)|\,dr \leqslant c(1+s^2)^{-q/2}.$$

Hence, collecting these estimates we obtain

$$|m_\Lambda * \psi(\lambda) - m_\Lambda(\lambda)| \leqslant c\Lambda^{-k}(1+\lambda^2)^{-q/2}.$$

This implies that

$$\begin{aligned}
&\left|\left| \sum_\lambda (m_\Lambda(\lambda) - m_\Lambda * \psi(\lambda))\mathscr{F}f(\lambda)\phi_\lambda(x) \right|\right|_{\mathbb{W}_h^2(\Omega)} \\
&\quad = \left\{ \sum_\lambda (1+\lambda^2)^h |m_\Lambda(\lambda) - m_\Lambda * \psi(\lambda)|^2 |\mathscr{F}f(\lambda)|^2 \right\}^{1/2} \\
&\quad \leqslant c\Lambda^{-k} \left\{ \sum_\lambda (1+\lambda^2)^{h-q} |\mathscr{F}f(\lambda)|^2 \right\}^{1/2}.
\end{aligned}$$

The desired estimate now follows from the assumed imbedding of $\mathbb{L}^1(\Omega)$ into $\mathbb{W}_{h-q}^2(\Omega)$ if $h-q$ is sufficiently negative. This proves the first inequality of the lemma, the proof of the second is similar. □

Remark. As mentioned in the introduction, there is an analog of the above theorem for expansions in eigenfunctions of the Laplace operator with arbitrary boundary conditions. It suffices that the laplacian with the boundary conditions is self-adjoint and it makes sense to consider eigenfunction expansions. Indeed, in Lemma 1.2 the boundary condition $u(t,x)=0$ if $x\in\partial\Omega$ for solutions to the wave equation plays no role and it can be replaced by a different one. Similarly, Lemma 1.3 depends only on the imbedding of $\mathbb{L}^1(\Omega)$ into $\mathbb{W}^2_h(\Omega)$ if h is sufficiently negative.

The theorem has the following corollaries.

Corollary 1.4. *Let A be an open set with compact closure in Ω and let $1\leqslant p\leqslant +\infty$. Then the operator $\mathscr{M}_\Lambda$ is continuous from $\mathbb{L}^p(\Omega)$ into $\mathbb{L}^p(A)$, with norm uniformly bounded in Λ.*

Proof. The result for eigenfunction expansions follows from the corresponding one for classical trigonometric expansions. Indeed,

$$\left\{\int_A |\mathscr{M}_\Lambda f(x)|^p\,dx\right\}^{1/p}$$
$$\leqslant\left\{\int_A |\mathscr{M}_\Lambda f(x)-\mathbb{M}_\Lambda f(x)|^p\,dx\right\}^{1/p}+\left\{\int_A |\mathbb{M}_\Lambda f(x)|^p\,dx\right\}^{1/p}.$$

By the above theorem and the Sobolev imbeddings, $\mathscr{M}_\Lambda f(x)-\mathbb{M}_\Lambda f(x)$ is a continuous function, bounded on A by the norm of $f(x)$ in $\mathbb{L}^1(\Omega)$. Since A and Ω have finite measure, the boundedness of $\mathscr{M}_\Lambda-\mathbb{M}_\Lambda$ from $\mathbb{L}^p(\Omega)$ into $\mathbb{L}^p(A)$ follows. Moreover, the operator $\mathbb{M}_\Lambda$ is bounded on $\mathbb{L}^p(\mathbb{R}^n)$ and, a fortiori, from $\mathbb{L}^p(\Omega)$ into $\mathbb{L}^p(A)$. □

Corollary 1.5. *Let A be an open set with compact closure in Ω and let $h>0$, $k>0$, $1\leqslant p\leqslant +\infty$. Then given a finite linear combination of eigenfunctions $P(x)=\sum_{\lambda\leqslant\Lambda}\mathscr{F}P(\lambda)\phi_\lambda(x)$, there exists an entire function of exponential type $Q(x)=\int_{\{2\pi|\xi|\leqslant 2\Lambda\}}\mathbb{F}Q(\xi)\exp(2\pi i\xi\cdot x)\,d\xi$ such that*

$$\left\{\int_{\mathbb{R}^n}|Q(x)|^p\,dx\right\}^{1/p}\leqslant c\left\{\int_\Omega |P(x)|^p\,dx\right\}^{1/p},$$
$$||P(x)-Q(x)||_{\mathbb{W}^2_h(A)}\leqslant c\Lambda^{-k}\int_\Omega |P(y)|\,dy.$$

Vice versa, an entire function of exponential type Λ can be approximated in A by a linear combination of eigenfunctions of type 2Λ with an error dominated by $c\Lambda^{-k}$.

Proof. Let $\mathscr{M}_\Lambda$ and $\mathbb{M}_\Lambda$ be the operators associated to a smooth even multiplier $m(s)$, with $m(s)=1$ if $|s|\leqslant 1$ and $m(s)=0$ if $|s|\geqslant 2$. If $P(x)$ is a linear combination of eigenfunctions of type Λ, then $P(x)=\mathscr{M}_\Lambda P(x)$ and $Q(x)=\mathbb{M}_\Lambda P(x)$ is a function of exponential type 2Λ with the required approximation properties.

The proof of the vice versa is similar. Let $Q(x)=\mathbb{M}_\Lambda Q(x)$ be a function of exponential type Λ. Restrict $Q(x)$ to Ω and define $P(x)=\mathscr{M}_\Lambda(Q\chi_\Omega)(x)$.

Then

$$\begin{aligned}&\|P(x)-Q(x)\|_{\mathbb{W}_h^2(A)}\\&\quad\leqslant\|\mathscr{M}_\Lambda(Q\chi_\Omega)(x)-\mathbb{M}_\Lambda(Q\chi_\Omega)(x)\|_{\mathbb{W}_h^2(A)}+\|\mathbb{M}_\Lambda(Q\chi_\Omega-Q)(x)\|_{\mathbb{W}_h^2(A)}.\end{aligned}$$

The first summand is controlled by the above theorem. In order to control the second summand observe that, since all derivatives of the kernel $K_\Lambda(y)=\Lambda^n K(\Lambda y)$ associated to $\mathbb{M}_\Lambda$ have fast decay at infinity and $(Q\chi_\Omega-Q)(x)=0$ in Ω, all derivatives of $\mathbb{M}_\Lambda(Q\chi_\Omega-Q)(x)$ are small in A. Indeed if $1/p+1/q=1$,

$$\begin{aligned}&\left|\frac{\partial^\alpha}{\partial x^\alpha}\mathbb{M}_\Lambda(Q\chi_\Omega-Q)(x)\right|\\&\quad=\left|\int_{\mathbb{R}^n-\Omega}\frac{\partial^\alpha}{\partial x^\alpha}\Lambda^n K(\Lambda(x-y))Q(y)\,dy\right|\\&\quad\leqslant c\Lambda^{n+|\alpha|-k}\left\{\int_{\mathbb{R}^n-\Omega}|x-y|^{-kq}\,dx\right\}^{1/q}\left\{\int_{\mathbb{R}^n}|Q(x)|^p\,dx\right\}^{1/p}.\qquad\square\end{aligned}$$

2. Bernstein and Jackson theorems

As we have seen, linear combinations of eigenfunctions can be well approximated, at least locally, by entire functions of exponential type, and vice versa. Hence it is natural to conjecture that, at least locally, eigenfunctions and entire functions have the same approximation properties. Here, in particular, we consider Jackson and Bernstein type results for eigenfunction expansions, but we start by proving a local Jackson type theorem of approximation by entire functions of exponential type.

Theorem 2.1. *Let A and B be open sets with $\bar{A}\subset B$ and let h and k be non-negative integers. Then, given $f(x)$ in $L^p(\mathbb{R}^n)$ with $1\leqslant p\leqslant+\infty$, there exist functions $Q(x)$ entire of exponential type Λ with*

$$\begin{aligned}&\left\{\int_A|f(x)-Q(x)|^p\,dx\right\}^{1/p}\\&\quad\leqslant c\sup_{|y|\leqslant 1/\Lambda}\left\{\int_B\left|\sum_{j=0}^h(-)^j\binom{h}{j}f(x+jy)\right|^p dx\right\}^{1/p}+c\Lambda^{-k}\left\{\int_{\mathbb{R}^n}|f(x)|^p\,dx\right\}^{1/p}.\end{aligned}$$

Proof. As in the classical proof of the Jackson theorem, one can construct kernels $K_\Lambda(y)=\Lambda^n K(\Lambda y)$ of exponential type Λ with fast decay at infinity and such that

$$f(x)-K_\Lambda*f(x)=\int_{\mathbb{R}^n}K_\Lambda(y)\sum_{j=0}^h(-)^j\binom{h}{j}f(x+jy)\,dy.$$

Hence

$$\left\{\int_A |f(x)-K_\Lambda * f(x)|^p\,dx\right\}^{1/p} \leqslant \int_{\mathbb{R}^n} |K_\Lambda(y)| \left\{\int_A \left|\sum_{j=0}^{h} (-)^j \binom{h}{j} f(x+jy)\right|^p dx\right\}^{1/p} dy.$$

Now, the moduli of smoothness satisfy the trivial estimate

$$\left\{\int_A \left|\sum_{j=0}^{h} (-)^j \binom{h}{j} f(x+jy)\right|^p dx\right\}^{1/p} \leqslant 2^h \left\{\int_{\mathbb{R}^n} |f(x)|^p\,dx\right\}^{1/p}.$$

Moreover, as soon as all sets $A+jy$ with $0 \leqslant j \leqslant h$ are in B,

$$\left\{\int_A \left|\sum_{j=0}^{h} (-)^j \binom{h}{j} f(x+jy)\right|^p dx\right\}^{1/p} \leqslant (1+\Lambda|y|)^h \sup_{|y|\leqslant 1/\Lambda} \left\{\int_B \left|\sum_{j=0}^{h} (-)^j \binom{h}{j} f(x+jy)\right|^p dx\right\}^{1/p}.$$

Observe that if $A+jy$ for some $0 \leqslant j \leqslant h$ is not completely contained in B, then $|y| \geqslant \varepsilon > 0$. Hence, since $K_\Lambda(y)$ is rapidly decreasing, for every h we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} |K_\Lambda(y)| \left\{\int_A \left|\sum_{j=0}^{h} (-)^j \binom{h}{j} f(x+jy)\right|^p dx\right\}^{1/p} dy \\
&\leqslant \int_{\mathbb{R}^n} (1+\Lambda|y|)^h |K_\Lambda(y)|\,dy \sup_{|y|\leqslant 1/\Lambda} \left\{\int_B \left|\sum_{j=0}^{h} (-)^j \binom{h}{j} f(x+jy)\right|^p dx\right\}^{1/p} \\
&\quad + 2^h \int_{\{|y|\geqslant \varepsilon\}} |K_\Lambda(y)|\,dy \left\{\int_{\mathbb{R}^n} |f(x)|^p\,dx\right\}^{1/p} \\
&\leqslant c \sup_{|y|\leqslant 1/\Lambda} \left\{\int_B \left|\sum_{j=0}^{h} (-)^j \binom{h}{j} f(x+jy)\right|^p dx\right\}^{1/p} \\
&\quad + c\Lambda^{-k} \left\{\int_{\mathbb{R}^n} |f(x)|^p\,dx\right\}^{1/p}. \qquad \square
\end{aligned}$$

Theorem 2.2. *Let A and B be open sets with $\bar{A} \subset B$ and $\bar{B} \subset \Omega$, and let h and k be non-negative integers. Then, given $f(x)$ in $L^p(\Omega)$ with $1 \leqslant p \leqslant +\infty$, there exist finite* *eigenfunctions expansions* $P(x) = \sum_{\lambda \leqslant \Lambda} \mathscr{F}P(\lambda)\phi_\lambda(x)$ *with*

$$\left\{ \int_A |f(x) - P(x)|^p \, dx \right\}^{1/p} \leqslant c \sup_{|y| \leqslant 1/\Lambda} \left\{ \int_B \left| \sum_{j=0}^{h} (-)^j \binom{h}{j} f(x+jy) \right|^p dx \right\}^{1/p} + c\Lambda^{-k} \left\{ \int_\Omega |f(x)|^p \, dx \right\}^{1/p}.$$

Proof. By the above theorem, there exist functions $Q(x)$ entire of exponential type $\Lambda/2$ with the required approximation properties. Moreover, by Corollary 1.5, these entire functions can be approximated in the norm of $\mathbb{W}_s^2(A)$ by linear combinations of eigenfunctions $P(x)$ of type Λ with an error dominated by Λ^{-k}. Since the norm $\mathbb{W}_s^2(A)$ dominates the norm $\mathbb{L}^p(A)$ when s is large, we finally obtain

$$\begin{aligned} \left\{ \int_A |f(x) - P(x)|^p \, dx \right\}^{1/p} &\leqslant \left\{ \int_A |f(x) - Q(x)|^p \, dx \right\}^{1/p} + \left\{ \int_A |Q(x) - P(x)|^p \, dx \right\}^{1/p} \\ &\leqslant c \sup_{|y| \leqslant 1/\Lambda} \left\{ \int_B \left| \sum_{j=0}^{h} (-)^j \binom{h}{j} f(x+jy) \right|^p dx \right\}^{1/p} \\ &\quad + c\Lambda^{-k} \left\{ \int_\Omega |f(x)|^p \, dx \right\}^{1/p}. \qquad \square \end{aligned}$$

Theorem 2.3. *Let* $f(x)$ *be a function in* $\mathbb{L}^p(\Omega)$ *with* $1 \leqslant p \leqslant +\infty$, *also assume that for some positive constants* α *and* c *and every* $\Lambda > 0$ *there exists finite eigenfunctions expansions* $P(x) = \sum_{\lambda \leqslant \Lambda} \mathscr{F}P(\lambda)\phi_\lambda(x)$, *such that*

$$\left\{ \int_\Omega |f(x) - P(x)|^p \, dx \right\}^{1/p} \leqslant c\Lambda^{-\alpha}.$$

Then on every open set A *with compact closure in* Ω *the function* $f(x)$ *is Lipschitz of order* α, *that is if* $h > \alpha$,

$$\left\{ \int_A \left| \sum_{j=0}^{h} (-)^j \binom{h}{j} f(x+jy) \right|^p dx \right\}^{1/p} \leqslant c|y|^\alpha.$$

Proof. Let $P_j(x)$ be eigenfunctions expansions of type 2^j with

$$\left\{ \int_\Omega |f(x) - P_j(x)|^p \, dx \right\}^{1/p} \leqslant c2^{-\alpha j}.$$

In particular, $f(x) = P_0(x) + \sum_{j=0}^{+\infty} (P_{j+1}(x) - P_j(x))$. By Corollary 1.5 there exist functions $Q_j(x)$ of exponential type 2^{j+2} with

$$\left\{ \int_{\mathbb{R}^n} |Q_j(x)|^p \, dx \right\}^{1/p} \leqslant c \left\{ \int_\Omega |P_{j+1}(x) - P_j(x)|^p \, dx \right\}^{1/p} \leqslant c2^{-\alpha j},$$

$$||P_{j+1}(x) - P_j(x) - Q_j(x)||_{\mathbb{W}_h^2(A)} \leqslant c2^{-kj} \int_\Omega |P_{j+1}(x) - P_j(x)| \, dy \leqslant c2^{-(k+\alpha)j}.$$

We can write

$$f(x) = P_0(x) + \sum_{j=0}^{+\infty} Q_j(x) + \sum_{j=0}^{+\infty} (P_{j+1}(x) - P_j(x) - Q_j(x)).$$

$P_0(x)$ is analytic in Ω and, by Bernstein theorem in $\mathbb{R}^n$, $\sum_{j=0}^{+\infty} Q_j(x)$ is Lipschitz of order α in all $\mathbb{R}^n$. Finally $\sum_{j=0}^{+\infty} (P_{j+1}(x) - P_j(x) - Q_j(x))$ converges in $\mathbb{W}_h^2(A)$ and if h is suitably large functions in this space are Lipschitz of order α. We conclude that also $f(x)$ is Lipschitz of order α in A. □

We conclude with some remarks.

Remark. The term $\Lambda^{-k}\{\int_\Omega |f(x)|^p \, dx\}^{1/p}$ in our Jackson type theorem is necessary. Indeed, if $f(x)$ is a polynomial of degree $h-1$ then the differences $\sum_{j=0}^{h} (-)^j \binom{h}{j} f(x+jy)$ vanish, while $f(x) \neq P(x)$. On the other hand, when non-zero the modulus of smoothness is at least of the order of Λ^{-h}, hence if $h<k$ it dominates the term Λ^{-k}.

Remark. It is possible to give a localized version of our Bernstein type theorem. Let A and B be open sets with $\bar{A} \subset B$ and $\bar{B} \subset \Omega$. If for every $\Lambda > 0$ there exist $P(x) = \sum_{\lambda \leqslant \Lambda} \mathscr{F}P(\lambda)\phi_\lambda(x)$ with $\{\int_\Omega |P(x)|^p \, dx\}^{1/p} \leqslant c\Lambda^k$ and $\{\int_B |f(x) - P(x)|^p \, dx\}^{1/p} \leqslant c\Lambda^{-\alpha}$, then $f(x)$ is Lipschitz of order α in A. The proof follows from a localized version of Bernstein theorem on Euclidean spaces.

Remark. The above Jackson and Bernstein type theorems guaranty regularity or approximation not in all the domain Ω, but only in open sets at positive distance from the boundary $\partial\Omega$. The following examples show that indeed some restrictions seem necessary. The eigenfunctions of the Dirichlet problem in $\Omega = (0,1)$ are the trigonometric functions $\{\sqrt{2}\sin(\pi k x)\}_{k=1}^{+\infty}$ and the eigenfunction expansions are the classical sine Fourier series. Let $f(x)$ be Lipschitz of order α in $(0,1)$, that is $|f(x) - f(y)| \leqslant c|x-y|^\alpha$ if $0 < x, y < 1$. Then our Jackson theorem guaranties the existence of trigonometric polynomials with $\{\int_\varepsilon^{1-\varepsilon} |f(x) - \sum_{k=1}^{n} c_k \sin(\pi k x)|^p \, dx\}^{1/p} \leqslant cn^{-\alpha}$ for $\varepsilon > 0$, but the inequality with $\varepsilon = 0$ may fail. Indeed if true then, by the classical Bernstein's theorem, the odd extension of $f(x)$ should be Lipschitz of order α in $(-1,1)$ and this would impose extra conditions on $f(x)$ in a neighborhood of $x = 0$

and 1. To be more explicit, the function $f(x) = 1$ is smooth in $(0,1)$ but cannot be uniformly approximated in $(0,1)$ by sine functions which vanish at zero. More in general, $f(x) = x^{\alpha}(1-x)^{\alpha}$ is smooth in $(0,1)$ but the uniform approximation of this function has not better order than $n^{-\alpha}$.

References

[1] L. Brandolini, L. Colzani, Localization and convergence of eigenfunction expansions, J. Fourier Anal. Appl. 5 (1999) 431–447.
[2] D.I. Cartwright, K. Kucharski, Jackson's theorem for compact connected Lie groups, J. Approx. Theory 55 (1988) 352–359.
[3] R.A. DeVore, G.G. Lorentz, Constructive Approximation, Springer, Berlin, 1985.
[4] S. Giulini, Bernstein and Jackson theorems for the Heisenberg group, J. Austral. Math. Soc. 38 (1985) 241–254.
[5] L. Hörmander, The Analysis of Linear Partial Differential Operators, Springer, Berlin, 1985.
[6] P.I. Lizorkin, S.M. Nikol'skij, Approximation of functions on manifolds, in: Functions Spaces, Differential Operators and Nonlinear Analysis, Teubner Texte Mathematik, Vol. 133, Teubner, Leipzig, 1993, pp. 185–199.
[7] S.M. Nikol'skij, Approximation of Functions of Several Variables and Imbedding Theorems, Springer, Berlin, 1975.
[8] M.A. Pinsky, M. Taylor, Pointwise Fourier inversion: a wave equation approach, J. Fourier Anal. Appl. 3 (1997) 647–703.
[9] M.E. Taylor, Pseudodifferential Operators, Princeton University Press, Princeton, NJ, 1981.